Irrational inequalities

We have two situations:

1) $\sqrt{P(x)} < Q(x)$ is equivalent to:

$$P(x) > 0 \wedge Q(x) \geq 0 \wedge P(x) < Q^2(x)$$

2) $\sqrt{P(x)} > Q(x)$ is equivalent to:

$$\left[P(x) \geq 0 \wedge Q(x) < 0\right] \vee \left[P(x) > Q^2(x) \wedge Q(x) \geq 0\right]$$

Example 1: Solve : $\sqrt{x+6} < x-6$

Solution:

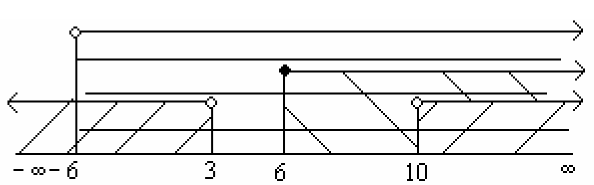
$$\sqrt{P(x)} < Q(x) \longrightarrow P(x) > 0 \wedge Q(x) \geq 0 \wedge P(x) < Q^2(x)$$

$$x+6 > 0 \wedge x-6 \geq 0 \wedge x+6 < (x-6)^2$$
$$x > -6 \wedge x \geq 6 \wedge x+6 < x^2 - 12x + 36$$

$$0 < x^2 - 12x + 36 - x - 6$$
$$0 < x^2 - 13x + 30$$

$$x^2 - 13x + 30 > 0$$
$$x^2 - 13x + 30 = 0$$
$$x_{1,2} = \frac{13 \pm \sqrt{169-120}}{2} = \frac{13 \pm 7}{2}$$
$$x_1 = 10$$
$$x_2 = 3$$
$$x \in (-\infty, 3) \cup (10, \infty)$$

When you solve all three inequalities, go to solution:



Cross-section of all the three solutions is : $x \in (10, \infty)$

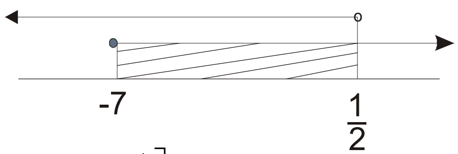
Example 2: Solve : $\sqrt{x+7} > 2x-1$

$$\sqrt{P(x)} > Q(x) \longrightarrow [P(x) \geq 0 \wedge Q(x) < 0] \vee [P(x) > Q^2(x) \wedge Q(x) \geq 0]$$

$$[x+7 \geq 0 \wedge 2x-1<0]$$

$$x \geq -7 \wedge 2x < 1$$

$$x < \frac{1}{2}$$



$$\left[x \geq -7 \wedge x < \frac{1}{2}\right] \qquad x \in \left[-7, \frac{1}{2}\right)$$

$\vee$

$$[x+7 > (2x-1)^2 \wedge 2x-1 \geq 0]$$

$$x+7 > 4x^2-4x+1 \quad \wedge x \geq \frac{1}{2}$$

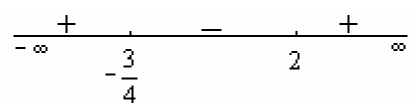
$$4x^2-4x+1-x-7<0$$

$$4x^2-5x-6<0$$

$$x_{1,2} = \frac{5 \pm 11}{8}$$

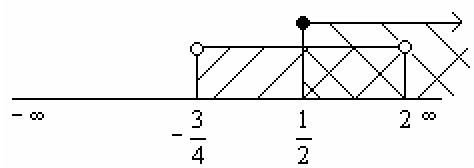
$$x_1 = 2$$

$$x_1 = -\frac{6}{8} = -\frac{3}{4}$$



$$x \in (-\frac{3}{4}, 2)$$

$$\left[x \in \left(-\frac{3}{4}, 2\right) \wedge x \geq \frac{1}{2}\right]$$



$$x \in [\frac{1}{2}, 2)$$

Solution is:

$$x \in [-7, \frac{1}{2}) \cup [\frac{1}{2}, 2)$$

$$\boxed{x \in [-7, 2)}$$